Lecture 20 ~~Other~~ Maximum Principles

Strong Principle For Subharmonic Functions

- A real-valued C^2 function u satisfying $-\Delta u \leq 0$ is called subharmonic. If $-\Delta u \geq 0$, it is superharmonic. Replacing u with $-u$ swaps between these cases

Recall: $u(x_0) = \frac{1}{A_n R^{n-1}} \int_{\partial B(x_0,R)} u(x)\,dS + \int_{B(x_0;R)} G_R(x-x_0)\Delta u(x)\,dx$ for $G_R < 0$ on $B(x_0,R)$

- Consider the case that u is subharmonic and let $x \in \Omega$ be a maximum of u. If $\overline{B(x,r)} \subseteq \Omega$ for some $r>0$,

$u(x) \leq \frac{1}{A_n r^{n-1}} \int_{\partial B(x,r)} u(y)\,dS(y)$,

but $u(y) \leq u(x)$ gives that $u(y) = u(x)$ ~~on~~ for $y \in \partial B(x,r)$. Replacing this with $u(x) \leq \frac{n}{A_n r^n} \int_{B(x,r)} u(y)\,dy$, we see that u must be constant in $B(x,r)$. Let $u \equiv M$ in $B(x,r)$

- This gives us intuition from the MVF: there are no "peaks" of u on the interior of Ω. We may extend this concept, using continuity and if we assume Ω is <u>connected</u>.

Let $E = \{y \in \Omega : u(y) < M\}$. Let $F = \{y \in \Omega : u(y) = M\}$. We know E is open because u is continuous: If $u(y_0) < M$, pick $\varepsilon = \frac{1}{2}(M - u(y_0))$ and for $|y - y_0| < \delta$, $|u(y) - u(y_0)| < \varepsilon$ means $u(y) < u(y_0) + \varepsilon < M$.

The argument above says F is open.

Now, recall that a connected set Ω <u>cannot</u> be written as a union of two open sets if the sets are disjoint and not empty (try doing it to $(0,1)$ if this is new to you).

- Indeed, recall that we defined connected to mean for $x_1, x_2 \in \Omega$ there exists a continuous

$p: [0,1] \to \Omega$ so $p(0) = x_1$, $p(1) = x_2$.

If $\Omega = E \cup F$, and $x_1 \in E$, $x_2 \in F$,

$p^{-1}(p([0,1]) \cap F) = \tilde{F}$ is open and in $[0,1]$

$p^{-1}(p([0,1]) \cap E) = \tilde{E}$ is open and in $[0,1]$.

Since $1 \in \tilde{F}$, Consider $\alpha \in \sup \tilde{E}$. Since $\tilde{E}$ is open $\alpha \notin \tilde{E}$. Further, $\tilde{E} \cup \tilde{F} = [0,1]$, so $\alpha \in \tilde{F}$.

However, then $(\alpha - \varepsilon, \alpha + \varepsilon) \subseteq \tilde{F}$ for some $\varepsilon > 0$, so $\sup(\tilde{E}) < \alpha$! Hence, we have a contradiction.

This long argument tells us the following: either E or F is empty, or...

$\boxed{\text{Th}^{\underline{m}}}$ Let $\Omega \subset \mathbb{R}^n$ be a domain. If $u \in C^2(\Omega; \mathbb{R}) \cap C^0(\bar{\Omega})$ is subharmonic, then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

and the maximum is attained at an interior point of Ω if and only if u is constant.

- For a superharmonic function, swapping signs gives the same statement with
$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u.$$

One of the most important results of the following:

$\boxed{\text{Corollary 9.6}}$ Suppose $u_1, u_2 \in C^2(\Omega) \cap C^0(\bar{\Omega})$ are solutions of $\Delta u = 0$ with

$$u_1|_{\partial\Omega} = g_1 \qquad u_2|_{\partial\Omega} = g_2$$

for $g_1, g_2 \in C^0(\partial\Omega)$. Then

$$\max_{\bar{\Omega}} |u_2 - u_1| \leq \max_{\partial\Omega} |g_2 - g_1|$$

In particular, the solution to the Laplace Eqn. is determined uniquely by bdry data.

Rmk: We may also prove Uniqueness by Energy Methods:

$$\int_{\Omega} \|\nabla u\|^2 dx = \int_{\partial\Omega} u \, \partial u/\partial n \, dS - 0 \qquad (\text{for } \Delta u = 0)$$

If $u = 0$ on $\partial\Omega$, $\nabla u = 0 \Rightarrow u \equiv 0$.

Weak Principle For Elliptic Equations

- On a domain $U \subseteq \mathbb{R}^n$, let

$$L = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_j(x) \frac{\partial}{\partial x_j} \qquad (E)$$

for $a_{ij}, b_j \in C^0(\Omega)$

let $a_{ij} = a_{ji}$ and, for each, $x \in U$

let $[a_{ij}(x)] = A(x)$ be a positive definite matrix.

~~By linearity, this says~~

For a maximum principle, we take an assumption of uniformity to the positive-definite matrix called uniform ellipticity:

there exists $K > 0$ so

$$\sum_{i,j=1}^{n} a_{ij}(x) v_i v_j \geq K \|v\|^2 \qquad \text{for } v \in \mathbb{R}^n, \; x \in U$$

$\boxed{\text{Th}^{\underline{m}}\ 9.7}$ Suppose $U \subseteq \mathbb{R}^n$ is a bdd. domain and L is an operator of the form (E) which is uniformly elliptic on U.

If $u \in C^2(\Omega; \mathbb{R}) \cap C^0(\bar{\Omega})$ satisfies

$$Lu \leq 0$$

in U, then

$$\max_{\bar{U}} u = \max_{\partial U} u$$

$\boxed{\text{Pf}}$ Let $w \in C^2(U; \mathbb{R})$ be arbitrary. Suppose w has a local max at x_0. Then, $\nabla w = 0$ at x_0, and

$$Lw = -\sum a_{ij}\, \partial x_i \partial x_j w$$

Since L is unif. elliptic <u>and</u> $[\partial x_i \partial x_j w]$ is negative-definite at x_0 (second derivative test), we will show that

$$Lw(x_0) \geq 0.$$

Indeed, let $A = [a_{ij}(x_0)]$, $B = [-\partial_{x_i}\partial_{x_j} w(x_0)]$. Notice A has a positive minimum eigenvalue λ_0, and we assume A to be diagonal (by a change-of-basis).

Then, $tr(AB) = \sum_{j=1}^{n} \lambda_j b_{jj} \geq \lambda_0 \, tr(B) \geq 0$

Since B is positive-semidefinite.

As $tr(AB) = Lw(x_0)$, $Lw(x_0) \geq 0$.

Apply this to $u(x)$. If we assume $Lu < 0$ on U, we notice a contradiction and obtain $\max_{\bar{U}} u = \max_{\partial U} u$.

Thus, we only need to relax our assumption to the case $Lu \leq 0$.

To do so, we alter u by a small approximation.

Pick $M > 0$ and set $h(x) = e^{Mx_1}$, so

$$Lh = [-a_{11}M^2 + b_1 M]h$$

As $a_{11} \geq \theta$ (uniform ellipticity), choosing $M > \frac{1}{\theta} \max_{\bar{U}} b_1$ gives $Lh < 0$.

Now, $L(u + \varepsilon h) < 0$ for $\varepsilon > 0$, so that

$$\max_{\bar{U}} u + \varepsilon h = \max_{\partial U} u + \varepsilon h.$$

For $h \geq 0$, $\max_{\bar{U}} u \leq \max_{\bar{U}} u + \varepsilon h$

Since U is bdd, $h(x) \leq e^{MR}$ for some large $R > 0$

and $\max_{\bar{U}} u \leq \left[\max_{\partial U} u\right] + \varepsilon e^{MR} \leq \max_{\bar{U}} u + \varepsilon e^{MR}$

As $\varepsilon \to 0$, we obtain our goal. $\square$

Application to the Heat Equation

- We noticed previously that heat tends to dissipate from a spatial maximum, suggesting that maxima either occur on the boundary or the earliest known time.

- There is a case of a mean value formula, though it is more opaque than for LaPlace's Equation. Instead, we approach as we did for general elliptic operators.

- First, let us define a "heat boundary"

$$\partial_h[(0,\tau)\times U] = (\{t=0\}\times U)\cup([0,\tau]\times\partial U)$$

and $C^{heat}(U) = \{ u \in C^0([0,\infty)\times\bar{U};\mathbb{R});\ u(\cdot,x)\in C^1((0,\infty)),$
$u(t,\cdot)\in C^2(U)\}$

Thm 9.8 Suppose $U\subset\mathbb{R}^n$ is a bdd. domain and $u\in C^{heat}(U)$ satisfies $(\partial_t-\Delta)u\le 0$ on $(0,\tau)\times U$. Then,

$$\max_{[0,\tau]\times\bar{U}} u = \max_{\partial_h[(0,\tau)\times U]} u$$

Pf Suppose u attains a max at $(t_0,x_0)\in(0,\tau)\times U$. Then,

$\partial u/\partial t(t_0,x_0)=0,\quad \nabla u(t_0,x_0)=0$ so that

$-\Delta u(t_0,x_0)\le 0$ by the heat equation.

Since $\partial^2 u/\partial x_j^2(t_0,x_0)\le 0$ (local max),

$-\Delta u(t_0,x_0)\ge 0.$

If this were strict, we'd be done. As above, we approximate this.

Set $\varepsilon>0$ and $u_\varepsilon = u + \varepsilon|x|^2$. Recall $\Delta|x|^2 = 2n$

So that $(\partial_t - \Delta)u_\varepsilon = \frac{\partial u}{\partial t} - \Delta u - 2n\varepsilon < 0.$

Then, u_ε attains its max on $\partial_h[(0,T)\times U] \cup (\{T\}\times U)$.

Let this point be $(t_\varepsilon, x_\varepsilon)$.

First, suppose $t_\varepsilon = T$ and $x_\varepsilon \in U$. We have

$$u_\varepsilon(t_\varepsilon, x_\varepsilon) = u_\varepsilon(T, x_\varepsilon) \geq u_\varepsilon(t, x_\varepsilon) \quad \text{for all } t \in [0,T]$$

so $\partial u_\varepsilon/\partial t\,(T, x_\varepsilon) \geq 0$ and so

$\Delta u_\varepsilon(T, x_\varepsilon) > 0$, a contradiction.

Thus, $(t_\varepsilon, x_\varepsilon) \in \partial_h[(0,T)\times U]$. Pick R so $|x|^2 < R$ on U and

$$\max_{[0,T]\times\bar{U}} u \leq \max_{[0,T]\times\bar{U}} u_\varepsilon \leq \left(\max_{\partial_h[(0,T)\times U]} u\right) + \varepsilon R$$

as $\varepsilon \to 0$,

$$\max_{[0,T]\times\bar{U}} u \leq 0 + \max_{\partial_h[(0,T)\times U]} u$$

as required $\square$

$\boxed{\text{Corollary 9.9}}$ Let $U \subseteq \mathbb{R}^n$ be a bdd domain. A solution of the heat equation $u \in C^{heat}(U)$ is uniquely determined by $u|_{\partial U}$ and $u|_{t=0}$.

$\boxed{\text{Pf}}$ Since u and $-u$ satisfy $(\partial_t - \Delta)(u) \leq 0$, $(\partial_t - \Delta)(-u) \leq 0$

$$\min_{\partial_h[(0,T)\times U]} u \leq u \leq \max_{\partial_h[(0,T)\times U]} u$$

Let u_1, u_2 be two solutions and $w = u_1 - u_2$ has $w|_{\partial_h[(0,T)\times U]} = 0$. The above gives $w \equiv 0$ on $[0,T]\times\bar{U}$.

- We can extend this result in two ways. First, we may reach $\partial_t u - Lu = 0$ for general elliptic operators. Second, we may find uniqueness on $\mathbb{R}^n$, which we do explicitly.

$\boxed{\text{Corollary 9.10}}$ Suppose that u is a classical solution to the heat equation $\partial u/\partial t - \Delta u = 0 \quad u|_{t=0} = g$

on $[0,\infty)\times\mathbb{R}^n$, and that u is bounded on $[0,\tau]\times\mathbb{R}^n$ for $\tau>0$. Then,

$$\max_{[0,\infty)\times\mathbb{R}^n} u \leq \max_{\mathbb{R}^n} g$$

$\boxed{\text{Pf}}$ Assume $u(t,x)\leq M$ on $[0,\tau]\times\mathbb{R}^n$. For $y\in\mathbb{R}^n$ and $\varepsilon>0$, set

$$v(t,x) = u(t,x) - \underbrace{\varepsilon(\tau-t)^{-n/2} e^{\frac{|x-y|^2}{4(\tau-t)}}}_{\text{Heat-kernel-ish}}$$

We may directly check $(\partial_t - \Delta)v = 0$ on $(0,\tau)\times\mathbb{R}^n$.

Pick $R>0$, and by the previous thm,

$$\max_{(0,\tau)\times B(y,R)} v \leq \max_{\partial_h((0,\tau)\times B(y,R))} v$$

By construction, $v(0,x)\leq g(x)$ and for $x\in\partial B(y,R)$,

$$v(t,x) \leq M - \varepsilon\tau^{-n/2} e^{R^2/4\tau}$$

then, for large R, $v(t,x)$ can't attain a max on $\partial B(y,R)$ so

$$\max_{(0,\tau)\times B(y,R)} v \leq \max_{B(y,R)} g \leq \max_{\mathbb{R}^n} g$$

or

$$u(t,y) \leq \max_{\mathbb{R}^n} g + \varepsilon(\tau-t)^{-n/2}$$

as $\varepsilon\to 0$, $u(t,y)\leq\max_{\mathbb{R}^n} g$ on $(0,\tau)\times\mathbb{R}^n$. Then, take $\tau\to\infty$. $\square$